Handout 1: The Levi-Civita connection

This handout derives an important formula for the connection in terms of derivatives of the metric which tells us how geometry determines the motion of particles in GR. This is for peace of mind only. *You do not need to remember this derivation or the equation* although you could be asked to apply it. Above all, it shows that the connection coefficients/Christoffel symbols are derivable from derivatives of the metric.

Remember first the definition of the connection coefficients:

$$\partial_\beta \vec{e}_\alpha = \Gamma^\sigma{}_{\alpha\beta}\, \vec{e}_\sigma,$$

where $\partial_\beta = \partial/\partial x^\beta$. Taking the scalar product of this with basis vector $\vec{e}_\delta$ gives

$$\vec{e}_\delta \cdot \partial_\beta \vec{e}_\alpha = \Gamma^\sigma{}_{\alpha\beta}\, \vec{e}_\delta \cdot \vec{e}_\sigma.$$

But by definition

$$\vec{e}_\delta \cdot \vec{e}_\sigma = g(\vec{e}_\delta, \vec{e}_\sigma) = g_{\delta\sigma},$$

so

$$\vec{e}_\delta \cdot \partial_\beta \vec{e}_\alpha = g_{\delta\sigma} \Gamma^\sigma{}_{\alpha\beta}.$$

The left-hand side is close to being the derivative of $\vec{e}_\delta \cdot \vec{e}_\alpha = g_{\delta\alpha}$. In fact

$$\partial_\beta g_{\delta\alpha} = \partial_\beta \vec{e}_\delta \cdot \vec{e}_\alpha + \vec{e}_\delta \cdot \partial_\beta \vec{e}_\alpha,$$

so that we can write

$$\partial_\beta g_{\delta\alpha} = g_{\alpha\sigma} \Gamma^\sigma{}_{\delta\beta} + g_{\delta\sigma} \Gamma^\sigma{}_{\alpha\beta}. \tag{1}$$

Cycling indices clockwise on the left-term, $\delta \to \beta$, $\alpha \to \delta$, $\beta \to \alpha$, gives

$$\partial_\alpha g_{\beta\delta} = g_{\delta\sigma} \Gamma^\sigma{}_{\beta\alpha} + g_{\beta\sigma} \Gamma^\sigma{}_{\delta\alpha}, \tag{2}$$

and cycling once more in the same sense gives

$$\partial_\delta g_{\alpha\beta} = g_{\beta\sigma} \Gamma^\sigma{}_{\alpha\delta} + g_{\alpha\sigma} \Gamma^\sigma{}_{\beta\delta}. \tag{3}$$

Adding Eqs 1 and 2 and taking away Eq. 3 gives:

$$\partial_\beta g_{\delta\alpha} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta} = g_{\alpha\sigma} \left(\Gamma^\sigma{}_{\delta\beta} - \Gamma^\sigma{}_{\beta\delta} \right) + g_{\delta\sigma} \left(\Gamma^\sigma{}_{\alpha\beta} + \Gamma^\sigma{}_{\beta\alpha} \right) + g_{\beta\sigma} \left(\Gamma^\sigma{}_{\delta\alpha} - \Gamma^\sigma{}_{\alpha\delta} \right).$$

In GR we can assume the connection to be *symmetric in its lower indices*, and so the right-hand side reduces to $2 g_{\delta\sigma} \Gamma^\sigma{}_{\alpha\beta}$. Contracting with $g^{\gamma\delta}/2$ then gives

$$\boxed{\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\delta\beta} - \partial_\delta g_{\alpha\beta} \right).}$$

This is the Levi-Civita connection. You may also see it written as

$$\boxed{\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta} \right),}$$

using the comma notation for partial derivatives. Therefore the connection is derivable from the metric.

Handout 2: Euler-Lagrange Equations

This handout provides some background on the calculus of variations leading to the Euler-Lagrange equations. This is for completeness and is not examinable. You should know how to *apply* these equations rather than memorise them.

The problem is to find a path which maximises or minimises the integral of a function L of the coordinates x^α and velocities $\dot{x}^\beta$ along a path parameterised by λ. This first arose in a famous problem set by Leibnitz and solved by Newton to find the shape of the curve which would lead to the minimum time for a bead sliding along it. "Velocities" are defined as derivatives of the coordinates with respect to λ. We want a path such that

$$\delta I = \delta \int L(x^\alpha, \dot{x}^\beta)\, d\lambda = 0.$$

α and β should be taken to reperesent all possible values.

Consider a small change in the path: δx^α, $\delta \dot{x}^\beta$, then

$$\delta L = \frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\beta} \delta \dot{x}^\beta.$$

The second term can be integrated by parts as follows:

$$\int \frac{\partial L}{\partial \dot{x}^\beta} \delta \dot{x}^\beta \, d\lambda = \left[\frac{\partial L}{\partial \dot{x}^\beta} \delta x^\beta \right] - \int \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\beta} \right) \delta x^\beta \, d\lambda.$$

If the start and end points are fixed ($\delta x^\beta = 0$), the first term on the right-hand side disappears and re-labelling β to α we are left with

$$\delta I = \int \left\{ \frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) \right\} \delta x^\alpha \, d\lambda.$$

For this to be true for arbitrary variations δx^α, we must have

$$\boxed{\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0,}$$

which are the Euler-Lagrange equations. Applied to the Lagrangian from lectures, $L = g_{\gamma\beta} \dot{x}^\gamma \dot{x}^\beta$, the Euler-Lagrange equations give

$$\frac{d}{d\lambda} \left(2 g_{\alpha\beta} \dot{x}^\beta \right) - \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} \dot{x}^\gamma \dot{x}^\beta = 0.$$

It is left as an exercise to show that this is equivalent to the equations of motion deduced from parallel transport

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0,$$

with $\Gamma^\alpha{}_{\beta\gamma}$ given by the Levi-Civita connection. Remember that dots indicate derivatives with respect to the affine parameter λ.

Handout 3: The Riemann Curvature tensor

Other than the key points at the bottom, the material of this handout is not examinable and you absolutely need not remember it! It is purely for your own satisfaction.

In lectures the Riemann curvature tensor was introduced via the expression

$$[\nabla_\gamma, \nabla_\beta]V_\alpha = V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta}.$$

Consider the left-hand term first:

$$V_{\alpha;\beta\gamma} = [V_{\alpha;\beta}]_{;\gamma} = V_{\alpha;\beta,\gamma} - \Gamma^\sigma{}_{\alpha\gamma} V_{\sigma;\beta} - \Gamma^\sigma{}_{\beta\gamma} V_{\alpha;\sigma}.$$

Expanding the covariant derivatives:

$$V_{\alpha;\beta\gamma} = \left(V_{\alpha,\beta} - \Gamma^\sigma{}_{\alpha\beta} V_\sigma\right)_{,\gamma} - \Gamma^\sigma{}_{\alpha\gamma}\left(V_{\sigma,\beta} - \Gamma^\rho{}_{\sigma\beta} V_\rho\right) - \Gamma^\sigma{}_{\beta\gamma}\left(V_{\alpha,\sigma} - \Gamma^\rho{}_{\alpha\sigma} V_\rho\right).$$

Swapping β and γ:

$$V_{\alpha;\gamma\beta} = \left(V_{\alpha,\gamma} - \Gamma^\sigma{}_{\alpha\gamma} V_\sigma\right)_{,\beta} - \Gamma^\sigma{}_{\alpha\beta}\left(V_{\sigma,\gamma} - \Gamma^\rho{}_{\sigma\gamma} V_\rho\right) - \Gamma^\sigma{}_{\gamma\beta}\left(V_{\alpha,\sigma} - \Gamma^\rho{}_{\alpha\sigma} V_\rho\right).$$

Subtracting the first from the second equation, and making use of the symmetry in the lower indices of the connection and the commutativity of partial differentiation, all the terms with derivatives of the $\vec{V}$ cancel leaving

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = \Gamma^\sigma{}_{\alpha\gamma,\beta} V_\sigma - \Gamma^\sigma{}_{\alpha\beta,\gamma} V_\sigma + \Gamma^\sigma{}_{\alpha\gamma}\Gamma^\rho{}_{\sigma\beta} V_\rho - \Gamma^\sigma{}_{\alpha\beta}\Gamma^\rho{}_{\sigma\gamma} V_\rho.$$

Re-labelling σ to ρ in the first term on the right-hand side leaves

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = \left(\Gamma^\rho{}_{\alpha\gamma,\beta} - \Gamma^\rho{}_{\alpha\beta,\gamma} + \Gamma^\sigma{}_{\alpha\gamma}\Gamma^\rho{}_{\sigma\beta} - \Gamma^\sigma{}_{\alpha\beta}\Gamma^\rho{}_{\sigma\gamma}\right) V_\rho.$$

Since the left-hand side is a tensor as is V_ρ, the term in brackets is a tensor too, the *Riemann curvature tensor* **R** with components:

$$\boxed{R^\rho{}_{\alpha\beta\gamma} = \Gamma^\rho{}_{\alpha\gamma,\beta} - \Gamma^\rho{}_{\alpha\beta,\gamma} + \Gamma^\sigma{}_{\alpha\gamma}\Gamma^\rho{}_{\sigma\beta} - \Gamma^\sigma{}_{\alpha\beta}\Gamma^\rho{}_{\sigma\gamma}.}$$

Key points:

- the equation for the Levi-Civita connection means that **R** contains the metric and its first derivatives
- the derivative of the connection means that **R** also contains *second derivatives* of the metric
- in freely-falling frames, the first derivatives of the metric are zero, but the second derivatives representing tidal forces do not vanish in general, so the curvature tensor and gravity cannot always be transformed away.

Handout 4: Properties of the Riemann tensor

This handout fills in some *non-examinable* background on the Riemann tensor. The fully covariant Riemann tensor components are:

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R^{\rho}{}_{\beta\gamma\delta} = g_{\alpha\rho} \left(\Gamma^{\rho}{}_{\beta\delta,\gamma} - \Gamma^{\rho}{}_{\beta\gamma,\delta} + \Gamma^{\sigma}{}_{\beta\delta} \Gamma^{\rho}{}_{\sigma\gamma} - \Gamma^{\sigma}{}_{\beta\gamma} \Gamma^{\rho}{}_{\sigma\delta} \right).$$

It is much simpler if we specialize to geodesic coordinates in which the connection (but not its derivatives) vanish:

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} \left(\Gamma^{\rho}{}_{\beta\delta,\gamma} - \Gamma^{\rho}{}_{\beta\gamma,\delta} \right) \tag{4}$$

If we substitute the connection from the Levi-Civita equation, we find

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma} \right).$$

The RHS is no longer a tensor, but it allows us to establish symmetries that *are* tensorial and therefore hold in all frames. The following symmetries are easily established:

$$\begin{aligned} &R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}, \\ &R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0. \end{aligned}$$

Bianchi identity: If one differentiates the Riemann tensor, one can show that

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} = 0.$$

This is the Bianchi identity, and it is important in calculating the divergence of the Ricci tensor needed for Einstein's field equations. To show this, contract α and δ, remembering that $R_{\beta\gamma} = g^{\alpha\delta} R_{\alpha\beta\gamma\delta}$, that $g_{\alpha\beta;\gamma} = 0$ and using the symmetries, which gives:

$$\begin{aligned} g^{\alpha\delta} \left(R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} \right) &= 0, \\ R_{\beta\gamma;\mu} + R^{\delta}{}_{\beta\mu\gamma;\delta} - R_{\beta\mu;\gamma} &= 0. \end{aligned}$$

The last line is the *contracted Bianchi indentity.* Contracting β and γ, raising and lowering indices where necessary, and again using the symmetries above then leads to

$$\begin{aligned} g^{\beta\gamma} \left(R_{\beta\gamma;\mu} + R^{\delta}{}_{\beta\mu\gamma;\delta} - R_{\beta\mu;\gamma} \right) &= 0, \\ R_{;\mu} - R^{\delta}{}_{\mu;\delta} - R^{\gamma}{}_{\mu;\gamma} &= 0. \end{aligned}$$

Changing $\delta \to \alpha$ and multiplying by $g^{\mu\beta}$ then shows that

$$R^{\alpha\beta}{}_{;\alpha} = \frac{1}{2} R_{;\mu} g^{\mu\beta},$$

a result used while justifying the field equations in lectures in order to balance the energy–momentum conservation equations $T^{\alpha\beta}{}_{;\alpha} = 0$. (Remember, since R is scalar, $R_{;\mu} = R_{,\mu}$.)

Handout 5: The Friedmann equations

This handout goes through the full derivation of the Friedmann equations, because it would be a shame to go through a whole GR course (or university physics course for that matter) without ever seeing how the equations that govern the entire Universe are derived. The algebra is lengthy however, and I would not expect you to reproduce it all in an exam. However, you could be asked for step 1 and parts of steps 2 and 4; only step 3 is entirely off-limits. Otherwise, the main purpose is to satisfy those for whom the phrase "it can be shown that" is irritating, but also to show why I had to resort to it.

I will follow the steps outlined in lectures.

Step 1: metric coefficients from the interval.

The FRW interval is

$$ds^2 = c^2\,dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right].$$

We first note from this that $g_{tt} = c^2$, $g_{rr} = -a^2/(1-kr^2)$, $g_{\theta\theta} = -a^2r^2$ and $g_{\phi\phi} = -a^2r^2\sin^2\theta$. Since the metric is diagonal then $g^{tt} = 1/g_{tt} = c^{-2}$, etc.

Step 2: connection from the metric.

The corresponding Lagrangian can be written down directly

$$L = c^2\dot{t}^2 - a^2(t)\left(\frac{\dot{r}^2}{1-kr^2} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right),$$

where the dots denote differentiation with respect to an affine parameter. Now apply the Euler-Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0.$$

For the t component:

$$2c^2\ddot{t} + 2aa'\left(\frac{\dot{r}^2}{1-kr^2} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) = 0,$$

where the prime as in a', denotes differentiation with respect to the Universal time, t. Therefore

$$\ddot{t} + \frac{aa'}{c^2}\left(\frac{\dot{r}^2}{1-kr^2} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) = 0. \tag{5}$$

Comparing with the geodesic equations of motion written as:

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\alpha\dot{x}^\beta = 0,$$

we deduce that

$$\Gamma^t{}_{rr} = aa'/c^2(1-kr^2),\quad \Gamma^t{}_{\theta\theta} = aa'r^2/c^2,\quad \Gamma^t{}_{\phi\phi} = aa'r^2\sin^2\theta/c^2.$$

The seven other independent components of the form $\Gamma^t{}_{\alpha\beta}$ are, mercifully, equal to zero.

The r-component of the Euler-Lagrange equations gives:

$$\frac{d}{d\lambda}\left[-\frac{2a^2\dot{r}}{1-kr^2}\right] - \left[-\frac{2ka^2r\dot{r}^2}{(1-kr^2)^2} - 2a^2r\dot{\theta}^2 - 2a^2r\sin^2\theta\,\dot{\phi}^2\right] = 0,$$

and so, using $da/d\lambda = (da/dt)\times(dt/d\lambda) = a'\dot{t}$,

$$-\frac{4aa'\dot{t}\dot{r}}{1-kr^2} - \frac{2a^2\ddot{r}}{1-kr^2} - \frac{4ka^2r\dot{r}^2}{(1-kr^2)^2} + \frac{2ka^2r\dot{r}^2}{(1-kr^2)^2} + 2a^2r\dot{\theta}^2 + 2a^2r\sin^2\theta\,\dot{\phi}^2 = 0.$$

Collecting terms and dividing out the coefficient of $\ddot{r}$ gives

$$\ddot{r} + \frac{2a'}{a}\dot{t}\dot{r} + \frac{kr\dot{r}^2}{1-kr^2} - r(1-kr^2)\dot{\theta}^2 - r(1-kr^2)\sin^2\theta\,\dot{\phi}^2 = 0. \tag{6}$$

Similarly the θ and ϕ components give

$$\ddot{\theta} + \frac{2a'}{a}\dot{\theta}\dot{t} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \tag{7}$$

$$\ddot{\phi} + \frac{2a'}{a}\dot{\phi}\dot{t} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0. \tag{8}$$

Reading the connection coefficients from Eqs. 6, 7 and 8, and adding in the t-components derived at the start, we find the following non-zero connection coefficients:

$$\begin{array}{lll}
\Gamma^t{}_{rr} = aa'/c^2(1-kr^2), & \Gamma^t{}_{\theta\theta} = aa'r^2/c^2, & \Gamma^t{}_{\phi\phi} = aa'r^2\sin^2\theta/c^2, \\
\Gamma^r{}_{tr} = a'/a, & \Gamma^r{}_{rr} = kr/(1-kr^2), & \Gamma^r{}_{\theta\theta} = -r(1-kr^2), \\
\Gamma^r{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta, & & \\
\Gamma^\theta{}_{t\theta} = a'/a, & \Gamma^\theta{}_{r\theta} = 1/r, & \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi{}_{t\phi} = a'/a, & \Gamma^\phi{}_{r\phi} = 1/r, & \Gamma^\phi{}_{\theta\phi} = \cot\theta.
\end{array}$$

(NB there are several errors in Hobson et al.'s version of these (1st edition, p377).) As a by-product, Eqs 5, 6, 7 and 8 are the geodesic equations of motion in the FRW metric, although note that since there is no explicit ϕ dependence, we could have gone straight for a first integral of Eq 8 from $\partial L/\partial\dot{\phi} =$ const:

$$a^2r^2\sin^2\theta\,\dot{\phi} = h.$$

Step 3: Ricci tensor from the connection.

The next stage is to work out the components of the Ricci tensor. This is where things get a bit hairy, and you may prefer to skim through this and cut to the chase by going to step 4. The Ricci tensor, which is the contraction of the Riemann tensor, is given by:

$$R_{\alpha\beta} = \Gamma^\rho{}_{\alpha\rho,\beta} - \Gamma^\rho{}_{\alpha\beta,\rho} + \Gamma^\rho{}_{\alpha\sigma}\Gamma^\sigma{}_{\rho\beta} - \Gamma^\rho{}_{\alpha\beta}\Gamma^\sigma{}_{\rho\sigma}.$$

Consider then the tt component. First set $\alpha = \beta = t$, and then expand out the summations over dummy indices ρ and σ, remembering that the commas indicate partial derivatives with respect to the associated coordinate indicated after the comma, and also that these become normal derivatives when they are time derivatives of $a(t)$:

$$\begin{aligned}
R_{tt} &= \Gamma^{\rho}{}_{t\rho,t} - \Gamma^{\rho}{}_{tt,\rho} + \Gamma^{\rho}{}_{t\sigma}\Gamma^{\sigma}{}_{\rho t} - \Gamma^{\rho}{}_{tt}\Gamma^{\sigma}{}_{\rho\sigma}, \\
&= 3\frac{d}{dt}(a'/a) + \Gamma^{\rho}{}_{t\sigma}\Gamma^{\sigma}{}_{\rho t}, \\
&= 3\frac{d}{dt}(a'/a) + \Gamma^{r}{}_{tr}\Gamma^{r}{}_{rt} + \Gamma^{\theta}{}_{t\theta}\Gamma^{\theta}{}_{r\theta} + \Gamma^{\phi}{}_{t\phi}\Gamma^{\phi}{}_{r\phi}, \\
&= 3\frac{d}{dt}(a'/a) + 3(a'/a)^2, \\
&= 3a''/a.
\end{aligned}$$

where the second line follows because there are three non-zero $\Gamma^{\rho}{}_{t\rho}$ components, all with the same value $= a'/a$, and because there are no components of the form $\Gamma^{\rho}{}_{tt}$. The contra-variant version of this can be calculated without evaluating any other components since the metric is diagonal so the only contravariant component of the metric with a t-index is $g^{tt} = c^{-2}$, so

$$R^{tt} = g^{t\rho}g^{t\sigma}R_{\rho\sigma} = g^{tt}g^{tt}R_{tt} = c^{-4}R_{tt} = \frac{3}{c^4}\frac{a''}{a}.$$

Next the rr component, now setting $\alpha = \beta = r$ in the general relation for the Ricci tensor and, as before, expanding out the summations over ρ and σ

$$\begin{aligned}
R_{rr} &= \Gamma^{\rho}{}_{r\rho,r} - \Gamma^{\rho}{}_{rr,\rho} + \Gamma^{\rho}{}_{r\sigma}\Gamma^{\sigma}{}_{\rho r} - \Gamma^{\rho}{}_{rr}\Gamma^{\sigma}{}_{\rho\sigma} \\
&= \Gamma^{r}{}_{rr,r} + \Gamma^{\theta}{}_{r\theta,r} + \Gamma^{\phi}{}_{r\phi,r} - \Gamma^{t}{}_{rr,t} - \Gamma^{r}{}_{rr,r} + \\
&\quad \Gamma^{t}{}_{rr}\Gamma^{r}{}_{tr} + \Gamma^{r}{}_{rt}\Gamma^{t}{}_{rr} + \Gamma^{r}{}_{rr}\Gamma^{r}{}_{rr} + \Gamma^{\theta}{}_{r\theta}\Gamma^{\theta}{}_{\theta r} + \Gamma^{\phi}{}_{r\phi}\Gamma^{\phi}{}_{\phi r} - \\
&\quad \Gamma^{r}{}_{rr}\Gamma^{r}{}_{rr} - \Gamma^{r}{}_{rr}\Gamma^{\theta}{}_{r\theta} - \Gamma^{r}{}_{rr}\Gamma^{\phi}{}_{r\phi} - \Gamma^{t}{}_{rr}\Gamma^{r}{}_{tr} - \Gamma^{t}{}_{rr}\Gamma^{\theta}{}_{t\theta} - \Gamma^{t}{}_{rr}\Gamma^{\phi}{}_{t\phi} \\
&= -\frac{2}{r^2} - \frac{\partial}{\partial t}\left[\frac{aa'}{c^2(1-kr^2)}\right] - \frac{a'^2}{c^2(1-kr^2)} + \frac{2}{r^2} - \frac{2k}{1-kr^2}, \\
&= -\frac{(aa'' + 2a'^2 + 2kc^2)}{c^2(1-kr^2)}.
\end{aligned}$$

The $\theta\theta$ component gives

$$
\begin{aligned}
R_{\theta\theta} &= \Gamma^{\rho}{}_{\theta\rho,\theta} - \Gamma^{\rho}{}_{\theta\theta,\rho} + \Gamma^{\rho}{}_{\theta\sigma}\Gamma^{\sigma}{}_{\rho\theta} - \Gamma^{\rho}{}_{\theta\theta}\Gamma^{\sigma}{}_{\rho\sigma} \\
&= \Gamma^{\phi}{}_{\theta\phi,\theta} - \Gamma^{t}{}_{\theta\theta,t} - \Gamma^{r}{}_{\theta\theta,r} + \\
&\quad \Gamma^{t}{}_{\theta\theta}\Gamma^{\theta}{}_{t\theta} + \Gamma^{r}{}_{\theta\theta}\Gamma^{\theta}{}_{r\theta} + \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{\theta}{}_{\theta r}\Gamma^{r}{}_{\theta\theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\phi\theta} - \\
&\quad \Gamma^{t}{}_{\theta\theta}\Gamma^{r}{}_{tr} - \Gamma^{t}{}_{\theta\theta}\Gamma^{\theta}{}_{t\theta} - \Gamma^{t}{}_{\theta\theta}\Gamma^{\phi}{}_{t\phi} - \Gamma^{r}{}_{\theta\theta}\Gamma^{r}{}_{rr} - \Gamma^{r}{}_{\theta\theta}\Gamma^{\theta}{}_{r\theta} - \Gamma^{r}{}_{\theta\theta}\Gamma^{\phi}{}_{r\phi}, \\
&= \Gamma^{\phi}{}_{\theta\phi,\theta} - \Gamma^{t}{}_{\theta\theta,t} - \Gamma^{r}{}_{\theta\theta,r} + \\
&\quad \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{\theta}{}_{\theta r}\Gamma^{r}{}_{\theta\theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\phi\theta} - \\
&\quad \Gamma^{t}{}_{\theta\theta}\Gamma^{r}{}_{tr} - \Gamma^{t}{}_{\theta\theta}\Gamma^{\phi}{}_{t\phi} - \Gamma^{r}{}_{\theta\theta}\Gamma^{r}{}_{rr} - \Gamma^{r}{}_{\theta\theta}\Gamma^{\phi}{}_{r\phi}, \\
&= \Gamma^{\phi}{}_{\theta\phi,\theta} - \Gamma^{t}{}_{\theta\theta,t} - \Gamma^{r}{}_{\theta\theta,r} + \\
&\quad \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{\theta}{}_{\theta r}\Gamma^{r}{}_{\theta\theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\phi\theta} - \\
&\quad \Gamma^{t}{}_{\theta\theta}\Gamma^{r}{}_{tr} - \Gamma^{t}{}_{\theta\theta}\Gamma^{\phi}{}_{t\phi} - \Gamma^{r}{}_{\theta\theta}\Gamma^{r}{}_{rr} - \Gamma^{r}{}_{\theta\theta}\Gamma^{\phi}{}_{r\phi}, \\
&= -\frac{1}{\sin^2\theta} - \frac{r^2(aa'' + a'^2)}{c^2} + 1 - 3kr^2 + \frac{a'}{a}\frac{r^2aa'}{c^2} - (1 - kr^2) + \cot^2\theta - \\
&\quad \frac{r^2aa'}{c^2}\frac{a'}{a} - \frac{r^2aa'}{c^2}\frac{a'}{a} + r(1-kr^2)\frac{kr^2}{1-kr^2} + r(1-kr^2)\frac{1}{r}, \\
&= -\frac{(aa'' + 2a'^2 + 2kc^2)}{c^2}r^2.
\end{aligned}
$$

Finally, the $\phi\phi$ component,

$$
\begin{aligned}
R_{\phi\phi} &= \Gamma^{\rho}{}_{\phi\rho,\phi} - \Gamma^{\rho}{}_{phi\phi,\rho} + \Gamma^{\rho}{}_{\phi\sigma}\Gamma^{\sigma}{}_{\rho\phi} - \Gamma^{\rho}{}_{\phi\phi}\Gamma^{\sigma}{}_{\rho\sigma} \\
&= -\Gamma^{t}{}_{\phi\phi,t} - \Gamma^{r}{}_{\phi\phi,r} - \Gamma^{\theta}{}_{\phi\phi,\theta} + \\
&\quad \Gamma^{t}{}_{\phi\phi}\Gamma^{\phi}{}_{t\phi} + \Gamma^{r}{}_{\phi\phi}\Gamma^{\phi}{}_{r\phi} + \Gamma^{\phi}{}_{\phi t}\Gamma^{t}{}_{\phi\phi} + \Gamma^{\phi}{}_{\phi r}\Gamma^{r}{}_{\phi\phi} + \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} + \Gamma^{\phi}{}_{\phi\theta}\Gamma^{\theta}{}_{\phi\phi} - \\
&\quad \Gamma^{t}{}_{\phi\phi}\Gamma^{r}{}_{tr} - \Gamma^{t}{}_{\phi\phi}\Gamma^{\theta}{}_{t\theta} - \Gamma^{t}{}_{\phi\phi}\Gamma^{\phi}{}_{t\phi} - \Gamma^{r}{}_{\phi\phi}\Gamma^{r}{}_{rr} - \Gamma^{r}{}_{\phi\phi}\Gamma^{\theta}{}_{r\theta} - \Gamma^{r}{}_{\phi\phi}\Gamma^{\phi}{}_{r\phi} - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi}, \\
&= -\Gamma^{t}{}_{\phi\phi,t} - \Gamma^{r}{}_{\phi\phi,r} - \Gamma^{\theta}{}_{\phi\phi,\theta} + \\
&\quad \Gamma^{\phi}{}_{\phi t}\Gamma^{t}{}_{\phi\phi} + \Gamma^{\phi}{}_{\phi r}\Gamma^{r}{}_{\phi\phi} + \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} - \Gamma^{t}{}_{\phi\phi}\Gamma^{r}{}_{tr} - \Gamma^{t}{}_{\phi\phi}\Gamma^{\theta}{}_{t\theta} - \Gamma^{r}{}_{\phi\phi}\Gamma^{r}{}_{rr} - \Gamma^{r}{}_{\phi\phi}\Gamma^{\theta}{}_{r\theta}, \\
&= -\frac{\partial}{\partial t}\left(\frac{r^2aa'\sin^2\theta}{c^2}\right) + \frac{\partial}{\partial r}\left(r(1-kr^2)\sin^2\theta\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right) \\
&\quad + \frac{r^2a'^2\sin^2\theta}{c^2} - (1-kr^2)\sin^2\theta - \cos^2\theta - \frac{r^2a'^2\sin^2\theta}{c^2} - \frac{r^2a'^2\sin^2\theta}{c^2} \\
&\quad + kr^2\sin^2\theta + (1-kr^2)\sin^2\theta, \\
&= -\frac{r^2\sin^2\theta(aa'' + a'^2)}{c^2} + (1 - 3kr^2)\sin^2\theta + \cos^2\theta - \sin^2\theta - \frac{r^2a'^2\sin^2\theta}{c^2} \\
&\quad -(1-kr^2)\sin^2\theta - \cos^2\theta + kr^2\sin^2\theta + (1-kr^2)\sin^2\theta \\
&= -\frac{(aa'' + 2a'^2 + 2kc^2)}{c^2}r^2\sin^2\theta
\end{aligned}
$$

Collecting all the components together and changing the derivatives with respect to time from primes to dots:

$$\begin{aligned}
R_{tt} &= \frac{3\ddot{a}}{a}, && (9)\\
R_{rr} &= -\frac{(a\ddot{a}+2\dot{a}^2+2kc^2)}{c^2(1-kr^2)}, && (10)\\
R_{\theta\theta} &= -\frac{(a\ddot{a}+2\dot{a}^2+2kc^2)}{c^2}r^2, && (11)\\
R_{\phi\phi} &= -\frac{(a\ddot{a}+2\dot{a}^2+2kc^2)}{c^2}r^2\sin^2\theta. && (12)
\end{aligned}$$

All other components are zero. The Ricci scalar follows from these

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 6\frac{(a\ddot{a}+\dot{a}^2+kc^2)}{c^2a^2}$$

This is, as expected, independent of spatial position.

Step 4: Substitute into the field equations.

As shown in lectures, the field equations can be written

$$R_{\alpha\beta} = -\frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right).$$

Assuming a perfect fluid

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)U_\alpha U_\beta - pg_{\alpha\beta},$$

and

$$T = g^{\alpha\beta}T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)g^{\alpha\beta}U_\alpha U_\beta - pg^{\alpha\beta}g_{\alpha\beta} = \rho c^2 - 3p.$$

In *comoving* coordinates, the fluid is stationary and so $U^\alpha = (1,0,0,0)$ and since $U_t = g_{tt}U^t$, then $U_\alpha = (c^2,0,0,0)$. Therefore

$$\begin{aligned}
T_{tt} &= \rho c^4,\\
T_{rr} &= \frac{a^2p}{1-kr^2},\\
T_{\theta\theta} &= a^2r^2p,\\
T_{\phi\phi} &= a^2r^2\sin^2\theta\, p.
\end{aligned}$$

Therefore the tt component of the field equations leads to

$$3\frac{\ddot{a}}{a} = -\frac{8\pi G}{c^4}\left(\rho c^4 - \frac{1}{2}(\rho c^2 - 3p)c^2\right),$$

which gives

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a, \qquad (13)$$

The rr component gives

$$-\frac{(a\ddot{a}+2\dot{a}^2+2kc^2)}{c^2(1-kr^2)} = -\frac{8\pi G}{c^4}\left(\frac{a^2p}{1-kr^2}-\frac{1}{2}(\rho c^2-3p)\frac{-a^2}{1-kr^2}\right),$$

and therefore

$$a\ddot{a}+2\dot{a}^2+2kc^2 = \frac{4\pi G}{c^2}(-p+\rho c^2)a^2.$$

Substituting for $\ddot{a}$ from Eq. 13 gives

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2. \tag{14}$$

The $\theta\theta$ and $\phi\phi$ components give the same relation.

From Eqs 13 and 14 one can also show the also are the Friedman equations. From them one can also show that

$$\dot{\rho} + \left(\rho + \frac{p}{c^2}\right)\frac{3\dot{a}}{a} = 0. \tag{15}$$

This equation comes directly from the energy conservations relation ${T^{\alpha\beta}}_{;\alpha} = 0$.

Eqs. 13, 14 and 15 are the Friedmann equations which govern the evolution of the size a and density ρ of the Universe. For summary they are

$$\begin{aligned}
\ddot{a} &= -\frac{4\pi G}{3}\left(\rho+\frac{3p}{c^2}\right)a,\\
\dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - kc^2,\\
\dot{\rho} &+ \left(\rho+\frac{p}{c^2}\right)\frac{3\dot{a}}{a} = 0.
\end{aligned}$$

Note that quasi-Newtonian derivations exist for the last two equations, but they are relativistic in origin and there is no explanation from Newtonian physics for the presence of pressure in the first equation or for the interpretation of k as spatial curvature. Moreover, one is forced to rely on arguments from GR to justify the neglect of the Universe when calculating the second equation. Nevertheless, the Newtonian derivations provide useful insight into the form of these equations. The first shows the decelleration of the expansion of the Universe owing to the gravitating matter/energy (and pressure) within it. The second has the form of a kinetic energy plus potential energy equals total energy equation, while the third says that the density of the Universe decreases because of the dilution of expansion (the $3\dot{a}/a$ term) and because of the work down by every element of volume in the Universe pushing back the rest of the Universe.